

Benford's law: a theoretical explanation for base 2

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Abstract

In this paper, we present a possible theoretical explanation for benford's law. We develop a recursive relation between the probabilities, using simple intuitive ideas. We first use numerical solutions of this recursion and verify that the solutions converge to the benford's law. Finally we solve the recursion analytically to yeild the benford's law for base 2.

1 Introduction

The leading significant digit of a random integer is one of $1, 2 \dots 9$. Intuitively, it is equally likely to be any of these nine figures. However, empirical observations, and the benford's law indicate the contrary. According to the law, the probability that a random integer, expressed in base 10, starts with the digit d is[1]

$$P_d = \text{Log}_{10}(1 + \frac{1}{d}) \quad (1)$$

$d = 1, 2 \dots 9$. This law was first proposed by newcomb in 1881[2]. It means, a random integer is most likely to start with 1, with a probability of 0.301, and least likely to start with 9, with a probability of 0.046. Note that the random integer is *unscaled*; i.e., it can be arbitrarily large. This is the suspected reason behind the nonuniform probabilities. On the other hand, if the random number was *scaled*, i.e, chosen from a bounded set, the corresponding probabilities are obtained through a direct calculation. For instance, consider a scale of 100, ie, the number is chosen from the set $[0, 100)$; the probabilities are indeed uniform. However, if the scale were 200, they would be nonuniform, with $d = 1$ acquiring a very large probability($> \frac{1}{2}$). In this paper, we use these scaled probabilities to arrive at the benford's values of unscaled probabilities. Before we proceed, we shall state the well known generalizations of benford's law.

The law is generalized to first two digits. The probability that a random integer starts with digits $d_1 d_2$ is given by

$$P_{d_1 d_2} = \text{Log}_{10}(1 + \frac{1}{d_2 + 10d_1}) = \text{Log}_{10}(1 + \frac{1}{d_1 d_2}) \quad (2)$$

It is further generalised to arbitrary number of significant digits, and expressed in an arbitrary base b as

$$P_{d_1 \dots d_k} = \text{Log}_b(1 + \frac{1}{d_k + bd_{k-1} + \dots + b^{k-1}d_1}) = \text{Log}_b(1 + \frac{1}{d_1 \dots d_k}) \quad (3)$$

where $d_1 \dots d_k$ is the number expressed in base b [4]. We shall consider the simple case of base 2. In the next section, we present the basic idea behind the proof, supported with examples and numerical calculations. The analytical proof is provided in section 3. We end with a brief discussion, in section 4.

2 Basic idea behind the proof and numerical estimates

Expressed in base 2, every number starts with 1. Hence we consider the first two significant digits, which are either 10 or 11. Let P_{10} and P_{11} be the corresponding probabilities. According to benford's law, $P_{10} = \text{log}_2(1 + \frac{1}{2}) = 0.5849625$, $P_{11} = \text{log}_2(1 + \frac{1}{3}) = 0.4150375$.

These are the unscaled probabilities. Unlike them, the scaled probabilities are easily evaluated. For instance, consider a scale of 1000; i.e, the random integer is chosen from the set $[0, 1000)$. Since, in this set, numbers starting from 10 and 11 are equally populated, the corresponding probabilities are $\frac{1}{2}$ each. This is true of any scale of the form $100 \dots 0$. Accordingly let us denote them by $P_{10}^{10} = P_{11}^{10} = \frac{1}{2}$. The superscript indicates that the scale is of the

form $10 \cdots 0$. Now consider a scale of 1100 . It can be verified that the probabilities are now $\frac{2}{3}$ and $\frac{1}{3}$. Also, this is true of any scale of the form $110 \cdots 0$. Let us denote them by $P_{10}^{11} = \frac{2}{3}$ and $P_{11}^{11} = \frac{1}{3}$.

Thus, the unscaled probability P_{10} is in between P_{10}^{10} and P_{10}^{11} , and P_{11} is inbetween P_{11}^{10} and P_{11}^{11} .

$$P_{10} = P_{10}^{10}w + P_{10}^{11}(1-w) \quad (4)$$

$$P_{11} = P_{11}^{10}w + P_{11}^{11}(1-w) \quad (5)$$

where w is the weight assosiated with the scale being of the form $100 \cdots 0$. To a first order, it can be approximated to the probability that a randomly chosen scale *starts with* 10 , which is P_{10} . Thus,

$$P_{10} = P_{10}^{10}P_{10} + P_{10}^{11}P_{11} \quad (6)$$

$$P_{11} = P_{11}^{10}P_{10} + P_{11}^{11}P_{11} \quad (7)$$

this gives $P_{10} = \frac{4}{7} = 0.57142$, and $P_{11} = \frac{3}{7} = 0.42857$. These are the first order approximations. The approximation lies in the assumption $w = P_{10}$; all integers starting from 10 are not of the form $10 \cdots 0$.

To sharpen the approximation, consider the first three significant digits. Using a similar notation, we denote the unscaled probabilities by P_{1xy} , where $x, y = 0, 1$. And the scaled probabilities by $P_{1xy}^{1\alpha\beta}$, $x, y, \alpha, \beta = 0, 1$. $P_{1xy}^{1\alpha\beta}$ is the probability that a random integer starts with $1xy$ when the scale is of the form $1\alpha\beta0 \cdots 0$. The equations, to the second approximation are

$$P_{1xy} = \sum_{\alpha\beta} P_{1xy}^{1\alpha\beta} P_{1\alpha\beta} \quad (8)$$

This is a set of four equations in four variables. Once we solve for $P_{1\alpha\beta}$, we can evaluate P_{10} using $P_{10} = P_{100} + P_{101}$.

To do this, we are to first evaluate $P_{1xy}^{1\alpha\beta}$, the population fraction of numbers starting from $1xy$ in the integer set $S = [0, 1\alpha\beta0 \cdots 0)$. This set can be broken in to three chunks $S = S_0 \cup S_1 \cup S_2$ where S_0, S_1 and S_2 are the integer sets,

$$\begin{aligned} S_0 &= [0, 1000 \cdots 0) \\ S_1 &= [1000 \cdots 0, 1\alpha00 \cdots 0) \\ S_2 &= [1\alpha00 \cdots 0, 1\alpha\beta0 \cdots 0) \end{aligned}$$

Note that they are disjoint. S_0 is the largest; S_1 is an enhancement over S_0 and S_2 is an enhancement over S_1 . If p_0, p_1 and p_2 are the population fractions of numbers starting from $1xy$ within the sets S_0, S_1 and S_2 respectively, we may write

$$P_{1xy}^{1\alpha\beta} = \frac{p_0|S_0| + p_1|S_1| + p_2|S_2|}{|S_0| + |S_1| + |S_2|} \quad (9)$$

where, $|S_j|$ is the number of elements in S_j . Clearly, $|S_1| = \frac{\alpha}{2}|S_0|$ and $|S_2| = \frac{\beta}{4}|S_0|$. In S_0 , the second and the third digits are equally distributed, i.e., $100, 101, 110, 111$ appear with equal populations. Hence $p_0 = \frac{1}{4}$. In S_1 , all numbers have second digit 0 and the third digit is equally distributed between 1 and 0 . So, $p_1 = \delta_{x0}\frac{1}{2}$. In S_2 , all numbers have second digit α , and third digit 0 . Therefore, $p_2 = \delta_{x\alpha}\delta_{y0}$. Thus,

$$P_{1xy}^{1\alpha\beta} = \frac{1 + \alpha\delta_{x0} + \beta\delta_{x\alpha}\delta_{y0}}{4 + 2\alpha + \beta} \quad (10)$$

The equation $P_{1xy} = \sum_{\alpha, \beta} P_{1xy}^{1\alpha\beta} P_{1\alpha\beta}$ reads

$$\begin{bmatrix} 1/4 & 2/5 & 1/3 & 2/7 \\ 1/4 & 1/5 & 1/3 & 2/7 \\ 1/4 & 1/5 & 1/6 & 2/7 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix} \begin{bmatrix} P_{100} \\ P_{101} \\ P_{110} \\ P_{111} \end{bmatrix} = \begin{bmatrix} P_{100} \\ P_{101} \\ P_{110} \\ P_{111} \end{bmatrix} \quad (11)$$

The solution, after normalizing the sum to 1 is

$$P_{100} = 0.3152$$

$$P_{101} = 0.2626$$

$$P_{110} = 0.22251$$

$$P_{111} = 0.1969$$

Using, $P_{100} + P_{101} = P_{10}$ and $P_{110} + P_{111} = P_{11}$, we obtain $P_{10} = 0.5778$, the second approximation. As expected, it is closer to the benford's value, 0.5849625, than the first approximation.

Higher order approximations can be obtained by considering a larger number of digits. Considering k digits after the first digit, the equation to be solved is a $2^k \times 2^k$ matrix equation

$$P_{1x_1 \dots x_k} = \sum_{\{\alpha_i\}} P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} P_{1\alpha_1 \dots \alpha_k} \quad (12)$$

where $P_{1x_1 \dots x_k}$ is the probability that an unscaled integer starts with $1x_1 \dots x_k$ and the matrix element, $P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k}$ is the corresponding probability with a scale of $1\alpha_1 \dots \alpha_k 0 \dots 0$. This can be evaluated easily. For values of k up to 10, they were solved numerically using python. Table -1 summarizes the results. The values suggest a neat convergence to the benford's value. Interestingly, the relative error falls exponentially. In the next section, we shall prove it analytically.

k	P_{10}	Rel err
1	0.571428	0.023
2	0.577861	0.012
3	0.581339	0.0062
4	0.583135	0.0031
5	0.584045	0.00156
6	0.584503	0.00078
7	0.584732	0.00039
8	0.584847	0.00019
9	0.584905	0.000097
10	0.584933	0.000049

Table 1: Estimates of P_{10} up to $k=10$. Value according to benford's law: $P_{10} = 0.584962$

3 Analytical Solution

In this section, we show that the benford's law is an exact solution to equation[12]. We are to solve the equation for $P_{1x_1 \dots x_k}$ in the limit of $k \rightarrow \infty$. And the matrix elements in this equation are evaluated in appendix A.

$$P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} = \frac{1 + Q_{\alpha x}}{2^k (1 + \alpha)} \quad (13)$$

We are to show that the solution is logarithmic, i.e., $P_{1x_1 \dots x_k} = \text{Log}[1 + \frac{1}{1x_1 \dots x_k}]$. Observe that this function has a first approximation of $\frac{1}{1x_1 \dots x_k}$, in the large k limit. Hence, we shall first show that this is a solution in the limit of large k . That is, we are to show, that

$$\frac{1}{(1 + x)} = \lim_{k \rightarrow \infty} \sum \frac{1}{2^k} \frac{1 + Q_{\alpha x}}{(1 + \alpha)^2} \quad (14)$$

x and α are numbers between 0 and 1 with k places. In the limit of $k \rightarrow \infty$, x and α are any real numbers between 0 and 1 and the sum is replaced by an integral

$$\frac{1}{(1 + x)} = \int_0^1 d\alpha \frac{1 + Q_{\alpha x}}{(1 + \alpha)^2} \quad (15)$$

We are to show the above relation. $Q_{\alpha x}$ is the sum of an infinite series. The integral is easily evaluated for each of these terms, and then summed up. The details of this proof has been completed in appendix B.

For a finite value of k , to evaluate $P_{1\beta_1 \dots \beta_k}$, we write it as

$$P_{1\beta_1 \dots \beta_k} = \sum_{\{\alpha_i\}} P_{1\beta_1 \dots \beta_k \alpha_1 \dots \alpha_l} \quad (16)$$

We have shown that in the large l limit,

$$\lim_{l \rightarrow \infty} P_{1\beta_1 \dots \beta_k \alpha_1 \dots \alpha_l} = \frac{1}{1\beta_1 \dots \beta_k \alpha_1 \dots \alpha_l} \quad (17)$$

Thus,

$$\begin{aligned}
P_{1\beta_1 \dots \beta_k} &= \lim_{l \rightarrow \infty} \sum_{\{\alpha_i\}} \frac{1}{1\beta_1 \dots \beta_k \alpha_1 \dots \alpha_l} = \lim_{l \rightarrow \infty} \sum_{n=0}^{2^l} \frac{1}{2^l (1\beta_1 \dots \beta_k) + n} \\
&= \ln \left(\frac{1\beta_1 \dots \beta_k + 1}{1\beta_1 \dots \beta_k} \right)
\end{aligned}$$

Normalizing, we obtain the benford's law

$$P_{1\beta_1 \dots \beta_k} = \log_2 \left(\frac{1\beta_1 \dots \beta_k + 1}{1\beta_1 \dots \beta_k} \right) \quad (18)$$

4 Discussion

So far, little light has been thrown in to the counterintuitive nature of benford's law. We haven't reconstructed our intuition so as to understand the law. The origin of the anomalous behaviour is still unclear. A strong reason why it is counterintuitive is that, the cardinalities of numbers starting from any digit is the same, and therefore we expect the probabilities to be the same as well. One step towards understanding it is to realise that, the probabilities measure the *occurrences* and not the *cardinalities*.

To understand it better, let $\{a_i\}$ be a sequence and $\{b_i\}$ be a sub sequence of $\{a_i\}$. For instance, let $a_i = i$ and $b_i = 2i$. a_i is the sequence of positive integers and b_i is the subsequence of even numbers. The probability that a randomly chosen element in $\{a_i\}$ is also an element in $\{b_i\}$ is $\frac{1}{2}$. Now, let $\{c_i\}$ be a subsequence of $\{b_i\}$, $c_i = 4i$, the sequence of multiples of four. The probability that a randomly chosen element in $\{a_i\}$ is also an element in $\{c_i\}$ is $\frac{1}{4}$. Even though $\{b_i\}$ and $\{c_i\}$ have the same cardinalities, and can be mapped to each other, the probabilities are not equal. In fact, the sequence $\{a_i\}$ can be rearranged such that every alternate term is an element of $\{c_i\}$.

$$\{a'_i\} : 1, 4, 2, 8, 3, 12, 5, 16, \dots$$

This sequence $\{a'_i\}$ is a rearrangement of $\{a_i\}$. The probability that a random element belongs to $\{c_i\}$ is now $\frac{1}{2}$. Hence, this probability is unrelated to the cardinality; instead, it is a measure of *frequency of occurrence* of the elements of $\{c_i\}$ in the parent sequence $\{a_i\}$. Hence, it changes on rearranging the parent sequence.

In the above examples, all the occurrences were periodic. Thus, even though the sequences were infinite, due to the periodicity, the calculation of the probability was as simple as it is in case of a finite set. However, in a benford sequence, there is no such periodicity, and therefore, the calculation is nontrivial. In this paper, we have outlined a possible analytical explanation for benford's law for base 2. It is very likely that, a similar strategy can yeild the law for any base. Therefore, further work in this direction is expected to be fruitful.

5 Appendices

5.1 Appendix A: Evaluating the matrix elements

In this appendix, we evaluate the coefficients $P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k}$. It is the population fraction of numbers starting from $1x_1 \dots x_k$ in the set $S = [0, 1\alpha_1 \dots \alpha_k 000 \dots 0]$. We shall use the same strategy again: break this set in to disjoint chunks.

$$S = [0, 100 \dots 0] \cup [100 \dots 0, 1\alpha_1 0 \dots 0] \cup \dots \cup [1\alpha_1 \dots \alpha_{k-1} 00 \dots 0, 1\alpha_1 \dots \alpha_k 0 \dots 0]$$

defining the sets,

$$S_0 = [0, 100 \dots 0] \quad \& \quad S_r = [1\alpha_1 \dots \alpha_{r-1} 00 \dots 0, 1\alpha_1 \dots \alpha_{r-1} \alpha_r 0 \dots 0]$$

we may write

$$S = S_0 \cup S_1 \cup \dots \cup S_k$$

Writing p_r =population fraction of numbers starting from $1x_1 \dots x_k$ in the set S_r and $|S_r|$ =size of S_r , we may write

$$P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} = \frac{p_0|S_0| + p_1|S_1| + \dots + p_k|S_k|}{|S_0| + |S_1| + \dots + |S_k|}$$

since the sets are disjoint. Clearly, $|S_r| = \frac{\alpha_r}{2^r} |S_0|$. And, in $|S_0|$, all numbers are equally populated, thus, $p_0 = \frac{1}{2^k}$. In the set S_r , all numbers have the first $r-1$ digits equal to $\alpha_1 \dots \alpha_{r-1}$ respectively, and the r^{th} digit is zero. The rest of the $k-r$ digits are 0 or 1 with a probability of $\frac{1}{2}$ each. Thus,

$$p_r = \delta_{\alpha_1 x_1} \delta_{\alpha_2 x_2} \dots \delta_{\alpha_{r-1} x_{r-1}} \delta_{0 x_r} \cdot \frac{1}{2^{k-r}}$$

Therefore,

$$P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} = \frac{1 + \alpha_1 \delta_{0x_1} + \alpha_2 \delta_{0x_2} \delta_{\alpha_1 x_1} + \dots + \alpha_k \delta_{0x_k} \delta_{\alpha_{k-1} x_{k-1}} \dots \delta_{\alpha_1 x_1}}{1\alpha_1 \dots \alpha_k}$$

where $1\alpha_1 \dots \alpha_k = 2^k + \alpha_1 2^{k-1} + \dots + \alpha_k$. We can express it conveniently in a better notation. Let us define

$$\alpha = \frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} + \dots + \frac{\alpha_k}{2^k} \quad \& \quad x = \frac{x_1}{2} + \frac{x_2}{2^2} + \dots + \frac{x_k}{2^k}$$

α and x are numbers between 0 and 1 with k places. In this notation,

$$P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} = \frac{1 + \alpha_1 \delta_{0x_1} + \alpha_2 \delta_{0x_2} \delta_{\alpha_1 x_1} + \dots + \alpha_k \delta_{0x_k} \delta_{\alpha_{k-1} x_{k-1}} \dots \delta_{\alpha_1 x_1}}{2^k (1 + \alpha)}$$

also, for brevity, define

$$\alpha_1 \delta_{0x_1} + \alpha_2 \delta_{0x_2} \delta_{\alpha_1 x_1} + \dots + \alpha_k \delta_{0x_k} \delta_{\alpha_{k-1} x_{k-1}} \dots \delta_{\alpha_1 x_1} = Q_{\alpha x}$$

so that

$$P_{1x_1 \dots x_k}^{1\alpha_1 \dots \alpha_k} = \frac{1 + Q_{\alpha x}}{2^k (1 + \alpha)}$$

5.2 Appendix B: Analytical Solution

In this appendix, we show that

$$\frac{1}{(1+x)} = \int_0^1 d\alpha \frac{1 + Q_{\alpha x}}{(1+\alpha)^2}$$

Note that the first term, after performing the integral is $\frac{1}{2}$. For convenience, let us make the substitution $t = 1 - x$; $t_r = 1 - x_r$. The integral corresponding to r^{th} term in $Q_{\alpha x}$ is given by

$$\int_0^1 \frac{d\alpha}{(1+\alpha)^2} t_r \alpha_r \delta_{\alpha_1 x_1} \delta_{\alpha_2 x_2} \dots \delta_{\alpha_{r-1} x_{r-1}}$$

t_r can be taken out. The delta terms inside fix the first r places of α . $\alpha_i = x_i = 1 - t_i$ up to $i = r-1$ and $\alpha_r = 1$. Thus, the integral can be written as:

$$t_r \int_{a_r}^{b_r} \frac{d\alpha}{(1+\alpha)^2} = t_r \left(\frac{1}{(1+a_r)} - \frac{1}{(1+b_r)} \right)$$

where, $[a_r, b_r]$ is the range in which none of the deltas inside are zero. This range is given by

$$a_r = 0.x_1 x_2 \dots x_{r-1} 1 = x^{[r-1]} + \frac{1}{2^r} = 1 - t^{[r-1]} - \frac{1}{2^r}$$

and

$$b_r = 0.x_1 x_2 \dots x_{r-1} 1111 \dots = 1 - t^{[r-1]}$$

where $t^{[r]}$ is the approximation of t up to r places;

$$t^{[r]} = \frac{t_1}{2} + \frac{t_2}{2^2} + \frac{t_3}{2^3} + \dots + \frac{t_r}{2^r}$$

Thus, the integral corresponding to the r^{th} term in $Q_{\alpha x}$ is

$$t_r \left(\frac{\frac{1}{2^r}}{(2 - t^{[r-1]})(2 - t^{[r-1]} - \frac{1}{2^r})} \right)$$

Thus, summing up, we obtain

$$\sum \frac{1}{2^k} \frac{1+Q_{\alpha x}}{(1+\alpha)^2} = \frac{1}{2} + \sum_{r=1}^{\infty} \frac{t_r}{2^r} \left(\frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{1}{2^r})} \right)$$

Next we show that the series on the RHS sums up to $\frac{1}{1+x}$ or $\frac{1}{2-t}$. Consider,

$$\frac{1}{2-t^{[r+1]}} - \frac{1}{2-t^{[r]}} = \frac{t^{[r+1]} - t^{[r]}}{(2-t^{[r]})(2-t^{[r+1]})} = \frac{t_{r+1}}{2^{r+1}} \frac{1}{(2-t^{[r]})(2-t^{[r]}-\frac{t_{r+1}}{2^{r+1}})}$$

Using the above repeatedly, we may expand $\frac{1}{2-t^{[r]}}$ as

$$\frac{1}{2-t^{[r]}} = \frac{1}{2} + \frac{t_1}{2(2-\frac{t_1}{2})} + \frac{t_2}{2^2} \frac{1}{(2-\frac{t_1}{2}-\frac{t_2}{4})(2-\frac{t_1}{2}-\frac{t_2}{4}-\frac{t_3}{8})} + \cdots + \frac{t_r}{2^r} \frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{t_r}{2^r})}$$

Further, since, t_r can take only two values, 0 and 1, we may write

$$\frac{t_r}{2^{[r]}} \frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{t_r}{2^r})} = \frac{t_r}{2^r} \frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{1}{2^r})}$$

Thus,

$$\frac{1}{2-t^{[r]}} = \frac{1}{2} + \frac{t_1}{2(2-\frac{1}{2})} + \frac{t_2}{2^2} \frac{1}{(2-\frac{t_1}{2}-\frac{1}{4})(2-\frac{t_1}{2}-\frac{t_2}{4}-\frac{1}{8})} + \cdots + \frac{t_r}{2^r} \frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{1}{2^r})}$$

And continuing the series,

$$\frac{1}{2-t} = \frac{1}{2} + \sum_{r=1}^{\infty} \frac{t_r}{2^r} \left(\frac{1}{(2-t^{[r-1]})(2-t^{[r-1]}-\frac{1}{2^r})} \right)$$

Thus,

$$\int_0^1 d\alpha \frac{1+Q_{\alpha x}}{(1+\alpha)^2} = \frac{1}{2-t} = \frac{1}{(1+x)}$$

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